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DESIGN OF CRYOCOOLERS FOR MICROWATT SUPERCONDUCTING DEVICES

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The primary applications of the cryocoolers considered here are for cooling various Josephson devices such as SQUID magnetometers and amplifiers, voltage standards, and microwave mixers and detectors. The common feature of these devices is their extremely low inherent bias power requirement, of the order of 10^{-7} W (or sometimes much less) per junction. This provides the possibility, not yet fully exploited of designing compact, low-power cryocoolers for these applications, the design criteria being totally different from those of any cryocoolers presently available. Several concepts have been explored and a number of laboratory model cryocoolers have been built. These include low-power non-magnetic regenerative machines of the Stirling or Gifford-McMahon type, three- or four-stage Joule-Thomson machines, liquid-helium dewars with integral small cryocoolers to reduce the evaporation rate, and liquid-helium dewars with integral continuously or intermittently operated small helium liquefiers to permit operation of cryogenic devices for indefinite time periods.

Key words: Cryocooler; cryogenics; refrigeration; superconducting devices; SQUID.

1. Introduction

A workshop on "Applications of Closed-Cycle Cryocoolers to Small Superconducting Devices"[1] was held at the National Bureau of Standards Boulder Laboratories in October 1977, out of which came the present series of conferences, more broadly conceived, on refrigeration for cryogenic sensors and electronic systems. The intent of this paper is to return to the narrower theme of the first workshop, to emphasize again the nature of the problem and to give at least the rudiments of a design philosophy which is appropriate to the needs of Josephson and other very low-power superconducting devices. This paper is presented in the conviction that the practical potential of superconducting devices, primarily SQUIDs and other Josephson devices, is largely unrealized precisely because of the lack of a compatible, low-cost cryocooler. The commercial viability of systems consisting of these devices and integral cryocoolers can hardly be judged from the market for the devices themselves, currently estimated to be of the order of a hundred per year, since the cost and inconvenience of the present cryogenic support systems greatly inhibits their use.

2. Small Superconducting Devices

For the past 20 years most cryoelectronic systems have been built around superconducting tunnel junctions or microbridges exhibiting the Josephson effect, commonly known as Josephson junctions. Bias power is typically in the range of a few microwatts (the order of 1 mA at 1 mV) down to a few picowatts per junction. Arrays of up to 1000 Josephson junctions should have bias power requirements of a milliwatt or less. While most Josephson devices are specifically designed for operation in the neighborhood of 4 K (that is, in liquid helium at atmospheric pressure or below), research and development continues on higher-temperature materials. Many niobium devices have been made which operate at temperatures of 8 to 9 K, and a double-junction Nb-Al-Ge SQUID

(the SQUID, or superconducting quantum interference device, used primarily as an ultra-sensitive magnetic sensor but also as memory element, counter, and for other purposes, consists of a low-inductance superconducting loop with one or two Josephson junctions appropriately biased for the application) has been demonstrated to operate at a temperature above 20 K [2].

Room-temperature transistors when used at low temperatures typically require much greater bias power than Josephson junctions. GaAs FETs which have been used as low-noise preamplifiers for SQUIDs require 20 to 50 mW, so that an array of these could require a considerable refrigeration capacity. However, these are not cryogenic devices, and their noise performance does not improve below 50 K or so, at which temperature refrigeration is at least 12 times cheaper, so to speak, than at 4 K. It has been suggested that a true cryogenic transistor is one of the prime requirements, along with reliable and convenient miniature cryocoolers, for the widespread acceptance of cryogenic instrumentation in general. A number of cryogenic transistors have been suggested [3,4].

There is considerable interest in arrays of cooled infrared sensors both for astronomical studies, and more extensively, for military applications. Both the scale of the arrays and the requirement of an infrared window to the outside world may impose relatively large heat loads on the cooling mechanism. In any case, this application will not be considered in this paper, although some of the design principles discussed may be applicable.

3. Refrigeration Requirements

Since Josephson devices require virtually no bias power, the essential function of a cryocooler reduces to that of intercepting radiation and conduction heat leaks along electrical connections and mechanical supports from the room-temperature environment. Elementary arguments show that it is much more efficient, and also easier, to provide for these heat leaks by some optimum distribution of cooling capacity at several discrete temperatures, or by a continuous distribution of refrigeration, than by simply providing a large cooling capacity at the low-temperature end. This principle is well-known, and yet it is common practice to specify the cold-end cooling capacity of cryocoolers and to mention the refrigeration at higher temperatures almost as an afterthought, if at all, rather than the other way around. Lending weight to the argument is the fact that heat conduction in all the commonly used materials, both metallic alloys and non-metals, but excluding copper and other pure metals, increases monotonically with temperature (see figure 1) [5]. The fourth-power law of radiation is a more dramatic example of the same tendency.

Some typical magnitudes of low-temperature heat leaks are instructive. Radiant energy from a low-emissivity (0.1) surface of a radiation shield at 20 K is about 0.9 mW/m^2. As an example of heat leaks through support structures, it can be shown that a mass of 1 kg at a temperature of 10 K can be kinematically supported at 10 G loading inside a radiation shield at 20 K by cords or wires 10 cm long with heat leaks of 9 μW for nylon or 60 μW for stainless steel (these heat leaks were calculated using thermal conductivities from figure 1 and tensile strengths of 900 MPa for nylon and 1800 MPa for stainless steel). Thus, the heat leaks would still be small for shorter and thicker support structures, and most cryoelectronic devices have much less than 1 kg mass. The case of electrical leads is slightly more complicated. For bias power in the range of milliwatts or less, negligible heat leak can be achieved by the use of fine wires of low-conductivity alloy. Signal lines, on the other hand, may have to have low electrical resistance, and relatively high thermal conductance, to avoid degrading the signal-to-noise ratio. Nominally pure copper is widely used for this purpose, but since its conductivity may be 1000 times that of most alloys at low temperatures, it may introduce unnecessarily large heat leaks to the cryogenic device to which the signal lines are connected. We will return to the subject of signal lines after a discussion of refrigeration mechanisms.

4. Liquid Helium As a Refrigeration Mechanism

A liquid-helium bath provides a relatively small cooling capacity at the boiling point (the heat of vaporization, 83 J/mol at 4.2 K). The cold vapor provides additional cooling capacity, the specific heat at constant pressure, which is nearly uniformly distributed in temperature above 8 or 10 K ($\sim$ 21 J/mol·K). The total cooling capacity of the vapor, $\sim$ 6000 J/mol, is more than 70 times the heat of vaporization. In general, liquid helium does not provide an ideal distribution of refrigeration. As shown below and elsewhere, ideal cryogenic refrigeration systems for small superconducting devices will usually require monotonically increasing cooling capacity vs temperature, and only in rather special applications (perhaps to absorb the ohmic heat generated in high-current leads to a superconducting magnet) would the temperature-independent cooling capacity of helium provide an efficient refrigeration mechanism. This may be the essential reason for using liquid nitrogen or some other mechanism (see example below), along with liquid helium as a low-temperature cryogen, to provide additional cooling capacity at higher temperature.

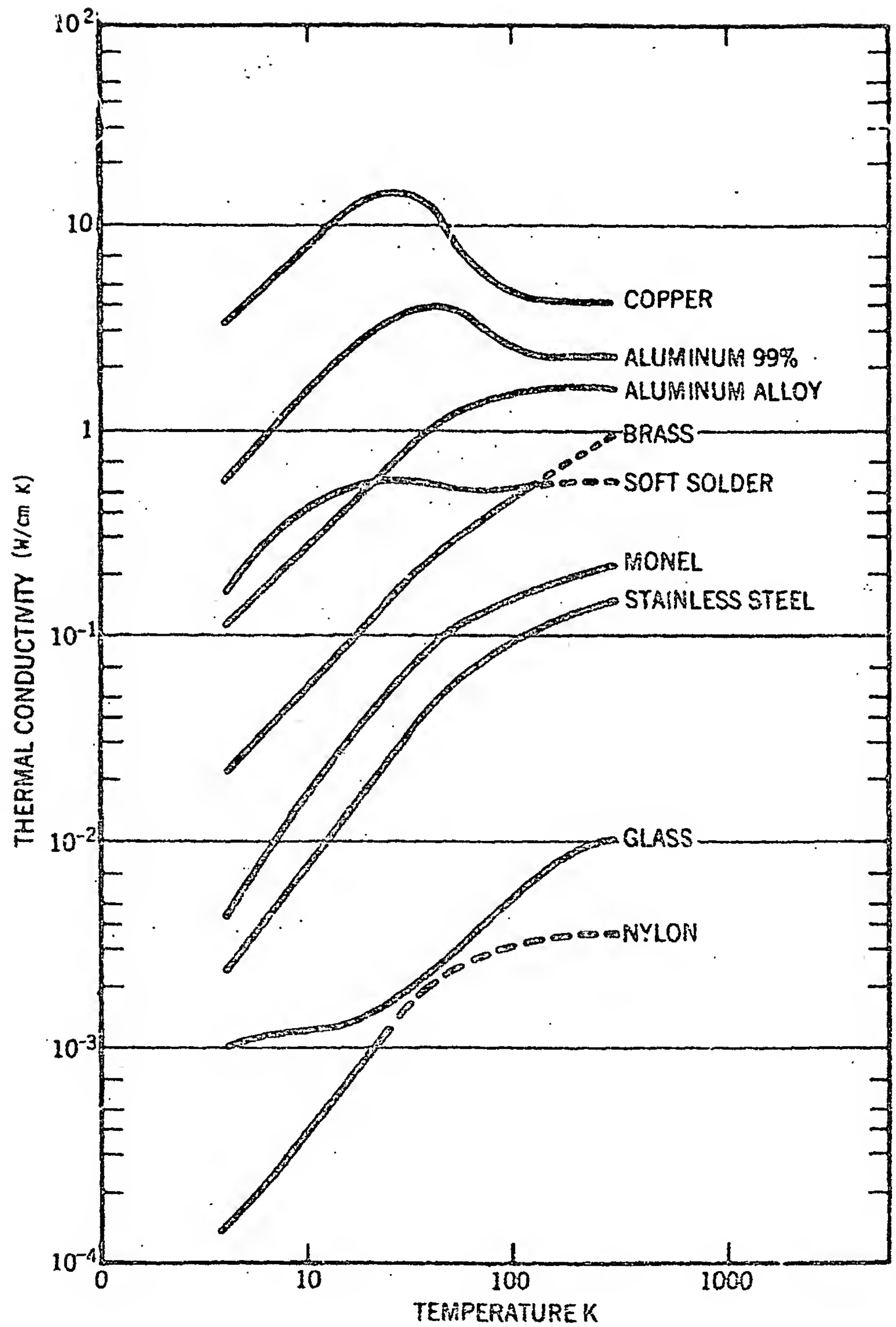

Figure 1. Thermal conductivity of some materials commonly used in cryogenic design.

5. Cryocoolers

Two types of cryocooler are under development for small superconducting devices, Joule-Thomson (J-T) and regenerative (Stirling, Gifford-McMahon, Vuilleumier). Joule-Thomson cryocoolers are highly attractive for very low-power applications because of their simplicity of construction and, more importantly, for having no solid moving parts in the cryogenic part of the system, so they generate essentially no magnetic interference or vibration (there may be high-frequency noise or hiss due to turbulent flow of the working fluid). Let us see how their qualitative features compare with the those of the heat leaks through various materials as noted above, and of thermal radiation. The following table gives the gross cooling capacities of a hypothetical four-stage Joule-Thomson cryocooler, with gas flow rates of 10 mL/s (standard temperature and pressure) per stage:

Working Fluid	CF_4	N_2	H_2	H_e
Temperature (K)	160	84	23	4.2
Low/high Pressures (MPa)	0.2/10	0.2/10	0.2/10	0.1/2
Gross Cooling Capacity (W)	3.3	1.1	0.13	0.012

The gross cooling capacities for a multi-stage reversible machine (Stirling, Carnot, etc.) with the same gas flows per stage are similar to the Joule-Thomson case. Here the cooling capacity, in the perfect-gas approximation, is of the form $PV(T2/T1)\ln(P1/P2)$, where the pressure-volume product PV is at standard temperature and pressure, P1/P2 is the compression ratio, and T2/T1 is the ratio of temperature of the stage to ambient temperature. Thus for $P = 0.1$ MPa (atmospheric pressure), $V = 10\ cm^3/s$ (per stage), $T1 = 300$ K, $\ln(P1/P2) = 1.0$, the gross cooling capacities for a hypothetical four-stage machine are:

Temperature (K)	150	50	15	5
Gross cooling capacity (W)	0.5	0.17	0.05	0.017

Using ordinary helium as the working fluid, the cooling capacity of the bottom stage may be considerably less than 0.017 W, since the perfect-gas approximation is quite inadequate at 5 K. These cooling capacities are roughly comparable to those of the Joule-Thomson example with the same flow rates, but note that Stirling and similar regenerative machines typically operate at much lower pressures and compression ratios than Joule-Thomson machines.

The point to be noted here is that in an intuitively "balanced" design (that is, comparable gas flows in all stages), the refrigeration capacities of the successive stages increase with increasing temperature, qualitatively matching the heat leaks that are expected from the low-conductivity materials commonly used in cryogenic design.

6. Electrical Connections

Copper, the most widely used material for electrical connections, may be a poor choice from the point of view of cryocooler design and thermal (Nyquist) noise generated in the lead resistance. The thermal conductivity of copper peaks at a temperature in the neighborhood of 20 K before dropping off linearly to zero at 0 K. Thus, the use of copper leads may result in large heat leaks just in the low-temperature region where cooling capacity is inherently small. The electrical resistivity of pure copper is small at low temperatures, so that the high-temperature portion of the leads will dominate the total lead resistance, and also the Nyquist noise. For most alloys, on the other hand, the thermal conductivity decreases more or less linearly below room temperature. It is easy to show that wires of practically any alloy such as brass, copper-nickel, or beryllium-copper, of the appropriate diameter for a specified total lead resistance, will reduce the heat leak at the low-temperature end by a factor of 100 or more, relative to copper, with only a moderate increase in the heat leak at the high-temperature end. The use of such alloys should provide a much better match to a "balanced" cryocooler design (as defined above) than pure copper. In addition, Nyquist noise generated in the leads will be lower in the alloy than in copper, since a relatively larger part of the electrical resistance will be at low temperature. The spectral density of total Nyquist noise power is proportional to the integral over the length of the wire of the product of Boltzmann's constant, the temperature, and the differential resistance, that is, the integral of $k_b T dR(T)$.

High-frequency and microwave leads can give large heat leaks if not specifically designed for cryogenic use. The above principles still apply, but since high-frequency currents flow only in a thin surface layer (less than 1 μm at 10 GHz in copper at room temperature), electrical losses and

heat leaks can both be reduced by using thin layers of low-resistivity metal on high-resistivity or insulating substrates, rather than using thick self-supporting conductors of low-resistivity metal. Furthermore, waveguides for very high frequency radiation can incorporate vacuum gaps to eliminate heat conduction entirely.

In a previous paper [6], we derived the optimum (minimum input power) distribution of refrigeration for cooling an electrical lead or a mechanical support of constant cross section, extending from ambient temperature T_{amb} to a low temperature T_o, whose thermal conductivity K could be expressed as a power of the temperature T: $K=K_nT^n$. The analysis was based on the Wiedemann-Franz Law for the relationship between thermal conductivity K and electrical resistivity ρ, that is, $K\rho/T \sim 2.4 \times 10^{-8}$ $W\text{-}\Omega/K^2$. Applying this analysis to the case of temperature-independent conductivity gives an optimum temperature distribution (see Appendix for a summary of the mathematical analysis)

$$T = T_o \exp(az), \text{ where } a = \ln(T_{amb}/T_o),$$

and z is the reduced distance measured from the cold end. For the case where the conductivity is proportional to temperature ($K = K_1T$), the optimum temperature distribution is

$$T = ((1-z)T_o^{1/2} + zT_{amb}^{1/2})^2.$$

It turns out that in all cases the optimum distribution of refrigeration has the same z-dependence as the temperature. In the earlier paper we gave, as an example, the ideal minimum input power required to refrigerate an electrical lead (or set of leads) with temperature-independent thermal conductivity and net electrical resistance of .023 Ω. The result was 315 mW for the distributed refrigeration plus 93 mW for the heat flow remaining at the cold end of the leads, a total of 408 mW of mechanical power to produce the required refrigeration. In the case of thermal conductivity proportional to temperature (and the same electrical resistance), the ideal minimum input power works out to be 280 mW for the distributed refrigeration and only 19 mW for the cold end, a total of 299 mW. Although the difference in total power for the two cases is hardly significant, the factor of five difference in the cold end term could be very significant for cryocoolers where the performance at the cold end is limited by regenerator losses and non-ideal gas properties. This numerical example provides some quantitative support for the qualitative arguments given above for choosing the right kind of materials for electrical connections.

The optimum temperature distribution of refrigeration is the quantity of interest in designing a cryocooler. The optimum distribution is temperature-independent for the case of constant conductivity and proportional to $T^{1/2}$ for the case of conductivity proportional to temperature. Thus, liquid helium would provide essentially ideal refrigeration for the improbable example where it is necessary to use only materials of constant conductivity (probably no such materials exist) and constant cross section. An additional degree of freedom available to the designer is to vary the cross section of the electrical wires or wave guides in some optimum way as a function of position within the cryocooler. In fact, deviations from the Wiedemann-Franz Law (see above) are such as to be favorable to the use of copper or other pure metals such as silver and unfavorable to the use of high-resistivity alloys such as stainless steel, provided the cross-section is varied in some more-or-less optimum way. Materials of moderate electrical resistivity such as brass obey the Wiedemann-Franz Law more closely. Consideration of all the options can make the optimum design a difficult analytical problem, but it can be asserted with some confidence (based partly on experience) that the casual use of pure copper for electrical leads in the low-temperature end of low-power cryoelectronic systems can be disastrous.

7. Examples

One notable example of the use of a cryocooler with small superconducting devices is the hybrid cryostat recently reported by Archer [7]. He incorporated a two-stage commercial cryocooler with a 4.5-liter helium reservoir to achieve better than a five-day operating time for a pair of 100-120 GHz receivers at 2.5 K. The receivers consist of superconducting (tunnel junction) mixers at 2.5 K and GaAs FET IF preamplifiers at 20 K. Estimated heat leaks (actual heat leaks were slightly greater) to the 2.5 K helium bath include radiation ($\sim$1.5 mW) and conduction ($\sim$12 mW) through fill tube, dc leads, IF coaxial lines, 3-mm waveguides, four mechanical tuning rods, and massive solid supports for the helium reservoir and the receivers. Direct-current and microwave-bias-power levels were not mentioned in the paper and can be assumed to be negligible. Heat leaks at higher temperatures are intercepted by the cryocooler, which provides heat sinks and radiation shields at 20 K and at 65 K. Heat conduction through electrical connections was minimized by using design principles and materials already described (see above): unplated stainless steel waveguides for local-oscillator power, vacuum gaps in the low-loss signal waveguides, copper-beryllium alloy for the coaxial lines, and brass wire for dc connections.

This example demonstrates the small magnitude of total heat leak that can be achieved by careful design in a rather difficult application. It should be noted that this impressive performance was obtained without taking advantage of two mechanisms which, in principle, might further reduce the 13.5 mW heat leak by a considerable factor. First, by thermally linking the evaporating helium vapor to the various conducting members listed above, most of the heat leak from 20 K could be intercepted before reaching the 2.5 K bath. Second, if optimally loaded support members in tension were substituted for the massive supports, much of the heat conduction could be eliminated, although this substitution might be inconvenient to put into practice. It is curious to note that with this system as described (or any similar system), the cryocooler can easily provide all of the refrigeration required at higher temperatures, so that only about 1% of the total cooling capacity of the liquid helium, namely the heat of vaporization at 2.5 K, is actually essential to the operation of the system. One percent is not as bad as it seems, however, since one must apply the Carnot factor in calculating the work necessary to perform this refrigeration. Needless to say, any mechanism which leads to reducing the required mass and size of the cold components (the helium reservoir, for example) has a synergistic effect of reducing heat leak through reduction or elimination of support structure and surface area. This might be significant if the helium reservoir were eliminated in favor of a microminiature Joule-Thomson or Stirling stage, perhaps using helium-3 as the working fluid, to provide continuous cooling at 2.5 K.

There are other examples of small superconducting devices being operated in hybrid cryostats or in cryocoolers, but few which illustrate so nicely the variety of design problems that may be encountered. One problem not encountered in this example is that of magnetic interference, which is of overriding concern in designing cryocoolers for SQUIDs and certain other Josephson devices. The problem of scaling Joule-Thomson refrigeration systems down to "microminiature" size, as needed for Josephson and similar devices, has been addressed by Little in previous conferences of this series. His work has resulted in the commercial production of a remarkable series of tiny J-T cryocoolers whose gas-flow channels are etched into the surface of glass plates which are then bonded to cover plates, giving milliwatt cooling capacities at 80 K when supplied with nitrogen at 10 or 20 MPa [8]. Multi-stage units capable of maintaining cryogenic temperatures required for superconducting devices have not yet been demonstrated, but such a development would be extremely interesting because of the possibility of integrating the superconducting circuits on the same substrate.

A four-stage J-T cryocooler using more conventional materials and techniques, and specifically intended for a SQUID biomedical gradiometer, is under development by Tward. It is described in another paper in these proceedings [9]. J-T systems are highly attractive for low-level magnetic-measurement devices because there are no solid moving parts in the cryogenic system itself, and so magnetic interference is inherently low or non-existent. Simple Stirling or Gifford-McMahon cryocoolers with gap regenerators, for temperatures in the range of 7 to 9 K, have been under development for several years in the author's laboratory and elsewhere [10 to 15]. These typically have four or five discrete stages or else a tapered displacer to provide a continuous distribution of refrigeration as anticipated by the discussion above. In either case, the available refrigeration capacity is very small by the usual standards, but sufficient for the purpose of cooling microwatt superconducting devices and their associated electrical connections and support structure, as anticipated by the discussion above. The problem of compressor contamination is inherently less serious with these machines than with Joule-Thomson machines, but nevertheless there is a similar need for extremely clean compressors or pressure-wave generators with the appropriate pressures and compression ratios. Temperatures as low as 4 K have not been achieved with regenerative machines operating at the same pressure throughout. However, the possibility of maintaining temperatures below 4 K has been demonstrated using a separate cold-end stage operating at sub-atmospheric pressure [16]. A J-T stage operating at the same peak pressure as the regenerative machine with which it is incorporated, so that only one compressor is required, has also been reported [17].

Perhaps the greatest challenge facing the designer of a practical miniature J-T cryocooler is to build a set of almost perfectly clean compressors to provide the necessary high pressures and high compression ratios for the different stages. Both Tward and Little (private communication), and others, are currently working on solutions to the problem. Metal bellows or diaphragms are attractive in principle, but not easy in practice, since the slightest plastic deformation of the metal (at the valve ports, for example) will cause almost immediate failure. Gas-lubricated or magnetically suspended clearance seals between piston and cylinder, and unlubricated sliding seals of glass or graphite-filled teflon on hardened metal, and "hard-on-hard" clearance seals where both piston and cylinder are made of hard materials like ceramic or metal carbides and nitrides have all been demonstrated, in various applications, with varying degrees of success and reliability. Yet another approach is to demand less in the way of cleanliness of the compressor and to effectively purify the gases after compression (see paper by Tward in these proceedings).

8. Discussion

Several laboratory cryocoolers for microwatt superconducting devices, based more or less on the refrigeration mechanisms and design principles summarized above, have been developed during the last few years. These have demonstrated significant advances in construction methods, use of materials, computer analysis, miniaturization, interference reduction, and efficiency. Although a completely satisfactory machine has not yet been produced, it is likely that this goal will be realized during the next year or two. Work in the immediate future will surely concentrate on the design of miniature, ultra-clean compressors. One or more of the current experimental cryocoolers will be used with SQUID magnetometers and gradiometers to determine levels of vibration and magnetic interference. Further miniaturization of Joule-Thomson systems and new concepts for integrating these with superconducting microcircuits will certainly inspire active interest.

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10. Appendix

Following are the results (from the analysis of reference 6) of the optimization of the refrigeration distribution along a cryogenic member (a wire or support post), of constant cross section A and length a, with one end at ambient temperature T_{amb} and the other end at a low temperature T_o, whose thermal conductivity can be expressed as a power of the temperature, $K = K_n T^n$. Here z is the fractional distance measured from the cold end, Q(z) is the heat flow rate, dQ/dz and dQ/dT are the optimum distributions of refrigeration in z and in T, respectively, required to achieve the minimum ideal mechanical input power W expended on the working fluid, and T(z) is the corresponding temperature distribution:

CASE 1: $K = \text{constant},\ (n = 0)$

$$T(z) = T_o e^{z\ln(T_{amb}/T_o)}$$

$$dQ/dz = (KA/a)T\ln^2(T_{amb}/T_o)$$

$$dQ/dT = (KA/a)\ln(T_{amb}/T_o)$$

$$W = (KA/a)T_o\ln^2(T_{amb}/T_o)$$

CASE 2: $K = K_n T^n,\ (n \neq 0)$

$$T(z) = (T_o^{n/2} + \alpha z)^{2/n} \quad \text{where } \alpha = T_{amb}^{n/2} - T_o^{n/2}$$

$$dQ/dz = 2\alpha^2(2 + n)K_n AT/an^2$$

$$dQ/dT = \alpha(2 + n)K_n AT^{n/2}/an$$

$$W = 4K_n A\alpha^2 T_{amb}/an^2$$